MODULE-IV

CHAPTER-4

(4.1, 4.2, 4.3, 4.4)

4.1 Power series method

Power series is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots\cdots$$

Since it contains infinite number of terms so it is called an infinite series.

where,

$a_m \rightarrow$ The co-efficient of the series

$x \rightarrow$ variable

$x_0 \rightarrow$ centre of the series.

In particular if $x_0 = 0$ then the series becomes

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$$

which is a power series with centre at 0.

Example

(i) $\sum_{m=0}^{\infty} 4^m (x+2)^m = 1 + 4(x+2) + 4^2(x+2)^2 + \cdots\cdots$

which is a power series with centre at -2

$a_0 = 1$, $a_1 = 4$, $a_2 = 4^2$ ---- and so on

(ii) $\sum_{m=0}^{\infty} (m+1)(x-1)^{2m} = 1 + 2(x-1)^2 + 3(x-1)^4 + \cdots\cdots$

which is a power series with centre at 1

$a_0 = 1$, $a_1 = 0$, $a_2 = 2$, $a_3 = 0$, $a_4 = 3$ --- and so on

(iii) $\sum_{m=0}^{\infty} (m+2)x^m = 2 + 3x + 4x^2 + \cdots\cdots$

which is a power series with centre at 0

$a_0 = 2$, $a_1 = 3$, $a_2 = 4$, ---- and so on,

Scanned with CamScanner

Scanned with CamScanner

Power series expansion of some known functions with centre at 0 (Maclaurin series)

(i) $\frac{1}{1-x} = (1-x)^{-1} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots\cdots$

(ii) $\frac{1}{1+x} = (1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - \cdots\cdots$

(iii) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$

(iv) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots$

(v) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$

Ordinary point and singular point

Let us consider the diffrential equation

$$P_0(x)\frac{dy}{dx} + P_1(x)y = 0 \quad \text{——— (1)}$$

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \quad \text{——— (2)}$$

For what value of x, $P_0(x) = 0$ that value of x is called a singular point. All other points are called ordinary point.

Example:-1

$$x\frac{dy}{dx} + y = 0$$

Here $P_0(x) = x$, we have $P_0(x) = 0$ at $x = 0$

So, All the points are ordinary points except $x = 0$

Thus $x = 0$ is a singular point.

Scanned with CamScanner

Scanned with CamScanner

Example:-2

$$(1-x^2)y'' - 2xy' + 3y = 0$$

Here, $P_0(x) = 1-x^2$, we have $P_0(x) = 0$ at $x = \pm 1$

So, all the points are ordinary points except $x = \pm 1$. So, $x = \pm 1$ are the singular point.

Example:-3

$$\frac{dy}{dx} + y = 0$$

Here, $P_0(x) = 1$, There is no value of x for which $P_0(x) = 0$. So. all the points are ordinary point ~~or~~ There is no singular point.

NOTE:

If $x = x_0$ is an ordinary point of the differential equation (1) or (2) then the power series solution is given by

$$y = \sum_{m=0}^{\infty} a_m (x-x_0)^m$$

Since one differential equation have infinite number of ordinary points so it has infinite number of power series solution

Example

$$(x+1)\frac{dy}{dx} + y = 0$$

All the points are ordinary points except $x = -1$.

Thus the ordinary points are

$\ldots, -3, -2, 0, 1, 2, \ldots$

So, for each ordinary point we can get one power series solution.

Scanned with CamScanner

Scanned with CamScanner

Ordinary point	Power series soln
0	$y = \sum_{m=0}^{\infty} a_m x^m$
1	$y = \sum_{m=0}^{\infty} a_m (x-1)^m$
2	$y = \sum_{m=0}^{\infty} a_m (x-2)^m$
-2	$y = \sum_{m=0}^{\infty} a_m (x+2)^m$
-3	$y = \sum_{m=0}^{\infty} a_m (x+3)^m$

& so on.

To solve a DE by power series method we need only one ordinary point if 0 is among the ordinary point then give first preference to 0 if 0 is not there than consider 1, -1, 2, -2, ··· etc.

How to solve a DE by using power series method

Let us consider the DE

$$\left.\begin{aligned} P_0(x)\frac{dy}{dx} + P_1(x)y = 0 \\ P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \end{aligned}\right\} \quad \text{---- (1)}$$

STEP:-1 Find out the ordinary point of equation (1)

If x_0 is an ordinary point then consider the power series solution as

$$y = \sum_{m=0}^{\infty} a_m (x-x_0)^m \quad \text{---- (2)}$$

Scanned with CamScanner

Scanned with CamScanner

$$y' = \sum_{m=1}^{\infty} m a_m (x-x_0)^{m-1}, \quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m (x-x_0)^{m-2}$$

STEP:-II

Put these values in equation (1)

STEP:-III

Equating the co-efficients of $(x-x_0)^m$ of both sides find out the values of the Co-efficients a_m.

STEP:-IV

Putting the values of the co-efficients in equation (2) we can get the power series Solution.

Problem Set - 4.1

NO-1-TO-12

Find the power series solution of the following DE.

2. $y' = ky$

$\Rightarrow y' - ky = 0$ ——— (1)

All the points are ordinary point of equation (1)

Clearly $x_0 = 0$ is an O.P

Let the P.S.S be

$$y = \sum_{m=0}^{\infty} a_m x^m \quad \text{——— (2)}$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

Putting these values in equation (1)

$$\sum_{m=1}^{\infty} m a_m x^{m-1} - k \sum_{m=0}^{\infty} a_m x^m = 0$$

Scanned with CamScanner

Scanned with CamScanner

$$\Rightarrow (a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots)$$
$$- K(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) = 0$$

$$\Rightarrow (a_1 - Ka_0) + (2a_2 - Ka_1)x + (3a_3 - Ka_2)x^2$$
$$+ (4a_4 - Ka_3)x^3 + \cdots = 0$$

Equating both sides.

$$a_1 - Ka_0 = 0 \Rightarrow a_1 = Ka_0$$

$$2a_2 - Ka_1 = 0 \Rightarrow a_2 = \frac{K}{2}a_1 = \frac{K}{2} \cdot Ka_0 = \frac{K^2}{2}a_0$$

$$3a_3 - Ka_2 = 0 \Rightarrow a_3 = \frac{K}{3}a_2 = \frac{K}{3} \cdot \frac{K^2}{2}a_0 = \frac{K^3}{6}a_0$$

$$4a_4 - Ka_3 = 0 \Rightarrow a_4 = \frac{K}{4}a_3 = \frac{K}{4} \cdot \frac{K^3}{6}a_0 = \frac{K^4}{24}a_0$$

and so on

Putting in equation (2) the P.S.S is

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$
$$= a_0 + Ka_0x + \frac{K^2}{2}a_0x^2 + \frac{K^3}{6}a_0x^3 + \frac{K^4}{24}a_0x^4 + \cdots$$
$$= a_0\left(1 + Kx + \frac{K^2}{2}x^2 + \frac{K^3}{6}x^3 + \frac{K^4}{24}x^4 + \cdots\right)$$
$$= \text{constant } a_0 e^{Kx}.$$

4. $(1-x)y' = y$

$$\Rightarrow (1-x)y' - y = 0 \quad \text{---------- (1)}$$

All the points are ordinary points of equation (1) except $x = 1$

Clearly $x_0 = 0$ is an ordinary point.

Let the power series solution be

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \text{---------- (2)}$$

Scanned with CamScanner

Scanned with CamScanner

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Putting these values in equation (1)

$$(1-x)\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow (a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots)$$
$$- (a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots)$$
$$- (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = 0$$

$$\Rightarrow (a_1 - a_0) + (2a_2 - 2a_1)x + (3a_3 - 3a_2)x^2$$
$$+ (4a_4 - 4a_3)x^3 + \cdots = 0$$

Equating both sides.

$$a_1 - a_0 = 0 \Rightarrow a_1 = a_0$$

$$2a_2 - 2a_1 = 0 \Rightarrow a_2 = a_1 = a_0$$

$$3a_3 - 3a_2 = 0 \Rightarrow a_3 = a_2 = a_0$$

& so on

Putting these values in equation (2) the power series soln is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + a_0 x + a_0 x^2 + a_0 x^3 + \cdots$$

$$= a_0 (1 + x + x^2 + x^3 + \cdots)$$

$$= a_0 (1-x)^{-1}.$$

Scanned with CamScanner

Scanned with CamScanner

12. $y'' = y$

$\Rightarrow y'' - y = 0$ ——— (1)

All the points are ordinary point of equation (1) clearly $x_0 = 0$ is an ordinary point.

Let the power series soln be

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \text{——— (2)}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Putting these values in eqn (1)

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$\Rightarrow (2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \cdots)$

$$- (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = 0$$

$\Rightarrow (2a_2 - a_0) + (6a_3 - a_1)x + (12a_4 - a_2)x^2$

$$+ (20 a_5 - a_3) x^3 + \cdots = 0$$

Equate both sides

$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{1}{2} a_0$

$6a_3 - a_1 = 0 \Rightarrow a_3 = \frac{1}{6} a_1$

$12 a_4 - a_2 = 0 \Rightarrow a_4 = \frac{1}{12} a_2 = \frac{1}{24} a_0$

$20 a_5 - a_3 = 0 \Rightarrow a_5 = \frac{1}{20} a_3 = \frac{1}{120} a_1$

& so on

Putting these values in equation (2) the P.S.S is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Scanned with CamScanner

Scanned with CamScanner

$$= a_0 + a_1 x + 2x^2 + 3x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_1 x^5 + \cdots$$

$$= a_0 \left(1 + \frac{1}{2} x^2 + \frac{1}{24} x^4 + \cdots\right) + a_1 \left(x + \frac{1}{6} x^3 + \frac{1}{120} x^5 + \cdots\right)$$

10. $y'' = y'$

$\Rightarrow y'' - y' = 0$ ——— (1)

Scanned with CamScanner

Scanned with CamScanner

Problem set-4.2

4.2 Theory of the power series method.

No-1-To-8

Find a power series solution.

3. $xy' - 3y = K$ (Const.) ——— (1)

Ans:- All the points are ordinary point of equation (1) except at $x=0$

Clearly $x_0 = 1$ is an o.p

Let the p.s.s be

$$y = \sum_{m=0}^{\infty} a_m (x-1)^m \quad \text{——— (2)}$$

$$y' = \sum_{m=1}^{\infty} m\, a_m (x-1)^{m-1}.$$

Putting these values in equation (1)

$$x \sum_{m=1}^{\infty} m\, a_m (x-1)^{m-1} - 3 \sum_{m=0}^{\infty} a_m (x-1)^m = K$$

$$\Rightarrow \{(x-1)+1\} \sum_{m=1}^{\infty} m\, a_m (x-1)^{m-1} - 3 \sum_{m=0}^{\infty} a_m (x-1)^m = K$$

$$\Rightarrow \sum_{m=1}^{\infty} m\, a_m (x-1)^m + \sum_{m=1}^{\infty} m\, a_m (x-1)^{m-1} - 3 \sum_{m=0}^{\infty} a_m (x-1)^m = K$$

$$\Rightarrow (a_1 (x-1) + 2a_2 (x-1)^2 + 3a_3 (x-1)^3 + 4a_4 (x-1)^4 + \cdots)$$

$$+ (a_1 + 2a_2 (x-1) + 3a_3 (x-1)^2 + 4a_4 (x-1)^3 + 5a_5 (x-1)^4 + \cdots)$$

$$-3 (a_0 + a_1 (x-1) + a_2 (x-1)^2 + a_3 (x-1)^3 + a_4 (x-1)^4 + \cdots) = K$$

$$\Rightarrow (a_1 - 3a_0) + (a_1 + 2a_2 - 3a_1)(x-1) + (2a_2 + 3a_3 - 3a_2)(x-1)^2 + (3a_3 + 4a_4 - 3a_3)(x-1)^3 + (4a_4 + 5a_5 - 3a_4)(x-1)^4 + \cdots = K$$

Equate both sides

$$a_1 - 3a_0 = K \Rightarrow a_1 = K + 3a_0$$

$$2a_2 - 2a_1 = 0 \Rightarrow a_2 = a_1 = K + 3a_0$$

$$3a_3 - a_2 = 0 \Rightarrow a_3 = \frac{1}{3}a_2 = \frac{1}{3}(K + 3a_0)$$

$$4a_4 = 0 \Rightarrow a_4 = 0$$

$$5a_5 + a_4 = 0 \Rightarrow a_5 = -\frac{1}{5}a_4 = -\frac{1}{5}(0) = 0$$

& so on.

Thus, the P.S.S from (2)

$$y = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4$$
$$+ a_5(x-1)^5 + \cdots\cdots$$

$$= a_0 + (K+3a_0)(x-1) + (K+3a_0)(x-1)^2$$
$$+ \frac{1}{3}(K+3a_0)(x-1)^3 + 0 + 0$$

$$= a_0 + Kx - K + 3a_0x - 3a_0 + Kx^2 - 2Kx + K$$
$$+ 3a_0x^2 - 6a_0x + 3a_0 + \frac{1}{3}(Kx^3 - 3x^2K + 3Kx - K + 3a_0x^3 - 9a_0x^2 + 3a_0x - 3a_0 +$$

$$= a_0 + Kx - K + 3a_0x - 3a_0 + Kx^2 - 2Kx + K$$
$$+ 3a_0x^2 - 6a_0x + 3a_0 + \frac{1}{3}Kx^3 - x^2K + Kx$$
$$- \frac{K}{3} + a_0x^3 - 3a_0x^2 + a_0x - a_0$$

Scanned with CamScanner

$$= a_0 + Kx - K + 3a_0 x - 3a_0 + (K+3a_0)(x^2 - 2x + 1)$$

$$+ \frac{1}{3}(K+3a_0)(x^3 - 3x^2 + 3x \mp 1)$$

$$= a_0 + Kx - K + 3a_0 x - 3a_0 + Kx^2 - 2Kx + K$$

$$+ 3a_0 x^2 - 6a_0 x + 3a_0 + \frac{1}{3}Kx^3 - Kx^2 + Kx \mp \frac{K}{3}$$

$$+ a_0 x^3 - 3a_0 x^2 + 3a_0 x \mp a_0$$

$$= -\frac{K}{3} + \frac{1}{3}Kx^3 + a_0 x^3$$

$$= -\frac{K}{3} + \left(\frac{1}{3}K + a_0\right)x^3$$

$$= -\frac{K}{3} + a_3 x^3 .$$

5. $y'' - 3y' + 2y = 0$ ——— (1)

All the powers are ordinary powers ∴ eqn (1) clearly $x_0 = 0$ is an o.p

Let the P.S.S be $y = \sum_{n=0}^{\infty} a_n x^n$ — (2)

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Putting these values in eq (1)

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 3 \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow (2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots)$$

$$-3 (a_1 + 2a_2 x + 3a_3 x^2 + \cdots)$$

$$+2 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = 0$$

$$\Rightarrow (2a_2 - 3a_1 + 2a_0) + (6a_3 - 6a_2 + 2a_1) x$$

$$+ \cdots = 0$$

Equating to zero

$$2a_2 - 3a_1 + 2a_0 = 0$$

$$\Rightarrow a_2 = \frac{3}{2} a_1 - a_0$$

$$6a_3 - 6a_2 + 2a_1 = 0 \Rightarrow a_3 = a_2 - \frac{1}{3} a_1$$

$$= \frac{3}{2} a_1 - a_0 - \frac{1}{3} a_1$$

$$= \frac{7}{6} a_1 - a_0$$

& so on

Putting in eq (2) the power series

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$\Rightarrow y = a_0 + a_1 x + \left(\frac{3}{2} a_1 - a_0\right) x^2$$

$$+ \left(\frac{7}{6} a_1 - a_0\right) x^3 + \cdots$$

8. Sol $(1-x^2)y'' - 2xy' + 2y = 0$ —— (1)

All the pts are o.p of eq (1) except

$x = \pm 1$

Clearly $x_0 = 0$ is an o.p

Let the p.s.s be

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— 2)}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Put the vals in eq (1)

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$-2 \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow (2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \cdots)$$

$$- (2a_2 x^2 + 6a_3 x^3 + 12 a_4 x^4 + \cdots)$$

$$-2 (a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + \cdots)$$

$$+ 2 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots) = 0$$

Scanned with CamScanner

$\Rightarrow (2\xi + 2c_0) + (6\xi - 2c_1 + 2c_1)x$

$+ (12c_4 - 2\xi - 4\xi + 2\xi)x^2 + (20c_5 - 6\xi - 6\xi$

$+ 2c_3)x^3 + \cdots = 0$

Equating coeff of

$2c_2 + 2c_0 = 0 \Rightarrow \xi = -c_0$

$6\xi = 0 \Rightarrow \xi = 0$

$12c_4 - 4\xi = 0 \Rightarrow c_4 = \frac{1}{3}\xi = -\frac{1}{3}c_0$

$20c_5 - 10\xi = 0 \Rightarrow c_5 = \frac{1}{2}\xi = 0.$

Thus the p.s.s is

$y = c_0 + c_1 x + \xi x^2 + \xi_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$

$= c_0 + c_1 x - c_0 x^2 + 0\,x^3 - \frac{1}{3}c_0 x^4 + 0x^5 + \cdots$

~~$= c_0 + c_1 \cdots$~~

$= c_0\left(1 - \frac{1}{2}x^2 - \frac{1}{3}x^4 - \cdots\right) + c_1 x.$

Q-To-If

Find the p.s.s in the form of $x-1$

or

Find the p.s.s by taking the ordinary point as 1.

Ex. $y'' - y = 0$ ——— (1)

All the powers are o.p of eqn (1)

Class no = 1 is an o.p.

Scanned with CamScanner

Let the p-s-s be

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n \quad \text{---} (2)$$

$$y' = \sum_{n=1}^{\infty} n\, a_n (x-1)^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n (x-1)^{n-2}$$

Putting these values in eq (1)

$$\sum_{n=2}^{\infty} n(n-1)\, a_n (x-1)^{n-2} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\Rightarrow \cancel{(2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \cdots)}$$

$$\Rightarrow (2a_2 + 6a_3(x-1) + 12a_4(x-1)^2 + 20a_5(x-1)^3 + \cdots)$$

$$- (a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \cdots) = 0$$

$$\Rightarrow (2a_2 - a_0) + (6a_3 - a_1)(x-1) + (12a_4 - a_2)(x-1)^2$$

$$+ (20a_5 - a_3)(x-1)^3 + \cdots = 0$$

Equating coeffs

$$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{1}{2} a_0$$

$$(6a_3 - a_1) = 0 \Rightarrow a_3 = \frac{1}{6} a_1$$

$$12a_4 - a_2 = 0 \Rightarrow a_4 = \frac{1}{12} a_2 = \frac{1}{24} a_0$$

$$20a_5 - a_3 = 0 \Rightarrow a_5 = \frac{1}{20} a_3 = \frac{1}{120} a_1 \text{ & so on}$$

Putting in eq (2) the p-s-s is

$$y = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + a_5(x-1)^5 + \cdots$$

$$= a_0 + a_1(x-1) + \frac{1}{2} a_0 (x-1)^2 + \frac{1}{6} a_1 (x-1)^3$$

$$+ \frac{1}{24} a_0 (x-1)^4 + \frac{1}{120} a_1 (x-1)^5 + \cdots$$

$$= a_0 \left(1 + \frac{1}{2}(x-1)^2 + \frac{1}{24}(x-1)^4 + \cdots\right) + a_1 \left((x-1) + \frac{1}{6}(x-1)^3 + \frac{1}{120}(x-1)^5 + \cdots\right)$$

Scanned with CamScanner

(11) $y' = \left(\frac{y}{x}\right) + 1$

$\Rightarrow xy' = y + x$ ——— (1)

All the points are O.P of eqn (1) except $x=0$

clearly $x=1$ is an O.P

Let the P.S.S be

$$y = \sum_{m=0}^{\infty} a_m (x-1)^m \quad \text{——— (2)}$$

$$y' = \sum_{m=1}^{\infty} m a_m (x-1)^{m-1}$$

Putting in eqn (1)

$$x \sum_{m=1}^{\infty} m a_m (x-1)^{m-1} - \sum_{m=0}^{\infty} a_m (x-1)^m = x$$

$$\Rightarrow [(x-1)+1] \sum_{m=1}^{\infty} m a_m (x-1)^{m-1} - \sum_{m=0}^{\infty} a_m (x-1)^m = x$$

$$\Rightarrow \sum_{m=1}^{\infty} m a_m (x-1)^m + \sum_{m=1}^{\infty} m a_m (x-1)^{m-1} - \sum_{m=0}^{\infty} a_m (x-1)^m = x$$

$$\Rightarrow (a_1 (x-1) + 2a_2 (x-1)^2 + 3a_3 (x-1)^3 + \cdots)$$

$$+ (a_1 + 2a_2 (x-1) + 3a_3 (x-1)^2 + 4a_4 (x-1)^3 + \cdots)$$

$$- (a_0 + a_1 (x-1) + a_2 (x-1)^2 + a_3 (x-1)^3 + \cdots) = x$$

$$\Rightarrow (a_1 - a_0) + (a_1 + 2a_2 - a_1)(x-1)$$

$$+ (2a_2 + 3a_3 - a_2)(x-1)^2 + \cdots = (x-1) + 1$$

Equating both sides

$$a_1 - a_0 = 1 \Rightarrow a_1 = 1 + a_0$$

$$2a_2 = 1 \Rightarrow a_2 = \frac{1}{2}$$

$$3a_3 + a_2 = 0 \Rightarrow a_3 = -\frac{1}{3}a_2 = -\frac{1}{6}$$

& so on.

Putting in eqn (2) the P.S.S :

$$y = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \cdots$$

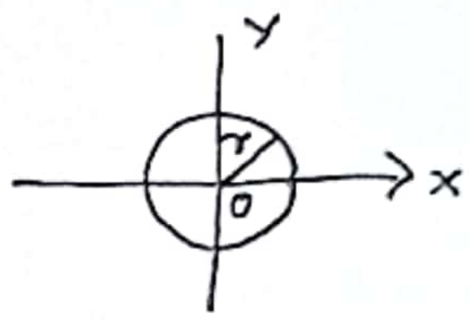
$$= a_0 + (1+a_0)(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \cdots$$

Scanned with CamScanner

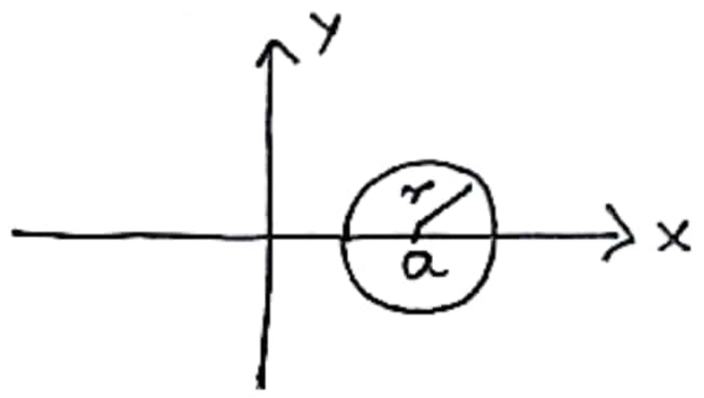
Radius of convergence

NOTE:-1

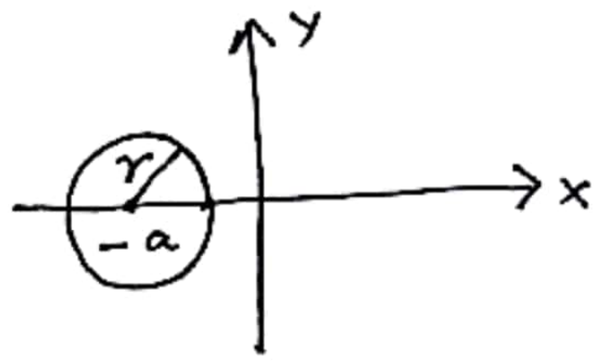
(i) $|x| = r$ → A circle with centre at 0 radius r



(ii) $|x-a| = r$ → A circle with centre at a radius r



(iii) $|x+a| = r$ → A circle with centre at $-a$ radius r



NOTE:-2

$|x| < r$
$|x-a| < r$
$|x+a| < r$

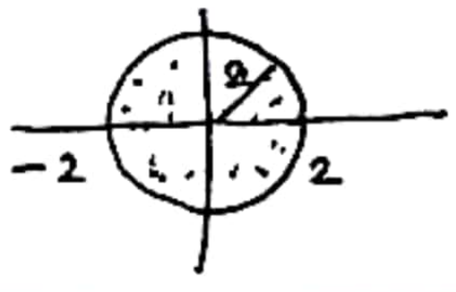
} means all the points inside the circle.

NOTE:-3

Suppose that a series is convergent for $|x| < 2$ that means if we draw a circle with centre at 0 and radius 2 inside the circle the series is defined. outside it is divergent or undefined.

At what radius the series is convergent that radius is called Radius of convergence it is denoted by R.

In this example $R = 2$



NOTE-4

(i) For, convergence

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} < 1 \qquad \text{(Cauchys Root test)}$$

(ii) For, Convergence,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \qquad \text{(D'Alembert's ratio test)}$$

~~NOTE~~: where a_n and a_{n+1} are the n^{th} and $(n+1)^{th}$ term of the series.

NOTE:-5

The first formula is applicable when all the terms present in a_n have power n or multiple of n.

The second formula is applicable in factorial & all other cases.

13-TO-21, Page-204, E.KREYSZIG (8TH EDITION)

Determine the radius of convergence of the following series.

20. $\sum_{n=0}^{\infty} n^n x^n$

Ans:- we have, the n^{th} term of the series

$$a_n = n^n x^n$$

For convergence,

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$$

$$\Rightarrow \lim_{n\to\infty} |n^n x^n|^{\frac{1}{n}} < 1$$

$$\Rightarrow \lim_{n\to\infty} |nx| < 1$$

Scanned with CamScanner

$$\Rightarrow \lim_{n\to\infty} |n| \lim_{n\to\infty} |x| < 1$$

$$\Rightarrow \lim_{n\to\infty} |x| < \frac{1}{\lim_{n\to\infty} |n|}$$

$$\Rightarrow \boxed{|x| < 0}$$

So, the series is convergesent for $|x| < 0$

The radius of convergence $R = 0$, Centre is 0.

17. $\sum_{n=0}^{\infty} \frac{(-1)^n}{k^n} x^{2n}$

Ans:- we have, the n^{th} term of the series

$$a_n = \frac{(-1)^n}{k^n} x^{2n}$$

For convergence,

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{(-1)^n}{k^n} x^{2n} \right|^{\frac{1}{n}} < 1$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{-x^2}{k} \right| < 1$$

$$\Rightarrow \left| \frac{x^2}{k} \right| < 1$$

$$\Rightarrow \frac{|x|^2}{|k|} < 1$$

$$\Rightarrow |x|^2 < |k|$$

$$\Rightarrow |x| < \sqrt{|k|}$$

So, the series is convergent for $|x| < \sqrt{|k|}$

The radius of convergence $R = \sqrt{|k|}$

Centre is 0.

Scanned with CamScanner

15. $\sum_{n=0}^{\infty} \frac{1}{3^n} (x-3)^{2n}$

Ans:- we have, the nth term of the series

$$a_n = \frac{1}{3^n} (x-3)^{2n}$$

For convergence,

$$\lim_{n \to \infty} |a_n|^{\frac{1}{n}} < 1$$

$$\Rightarrow \lim_{n \to \infty} \left| \frac{1}{3^n} (x-3)^{2n} \right|^{\frac{1}{n}} < 1$$

$$\Rightarrow \lim_{n \to \infty} \left| \frac{(x-3)^2}{3} \right| < 1$$

$$\Rightarrow \left| \frac{(x-3)^2}{3} \right| < 1$$

$$\Rightarrow \frac{|(x-3)^2|}{3} < 1$$

$$\Rightarrow |x-3|^2 < 3 \Rightarrow \boxed{|x-3| < \sqrt{3}}.$$

So, the series is convergent for $|x-3| < \sqrt{3}$

The radius of convergence $R = \sqrt{3}$

centre is 0.

18 $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

we have, the nth term of the series

$$a_n = \frac{x^{2n+1}}{(2n+1)!} \qquad a_{n+1} = \frac{x^{2n+3}}{(2n+3)!}$$

For convergence,

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

Scanned with CamScanner

$$\Rightarrow \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \times \frac{(2n+1)!}{x^{2n+1}} \right| < 1$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+3)} \right| < 1$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+3)} \right| \lim_{n\to\infty} |x^2| < 1$$

$$\Rightarrow \lim_{n\to\infty} |x|^2 < \frac{1}{\lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+3)} \right|}$$

$$\Rightarrow |x|^2 < \infty$$

$$\Rightarrow |x| < \infty$$

So, the series is convergent for $|x| < \infty$

The radius of convergence $R = \infty$

centre is 0.

14. $\sum_{n=0}^{\infty} (n+1)\, n\, x^n$

we have, the n^{th} term of the series

$$a_n = (n+1)\, n\, x^n$$

$$a_{n+1} = (n+2)(n+1)\, x^{n+1}$$

For convergence,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{(n+2)(n+1)\, x^{n+1}}{(n+1)\, n\, x^n} \right| < 1$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{(n+2)\, x}{n} \right| < 1$$

Scanned with CamScanner

$$\Rightarrow \lim_{n\to\infty} \left|\frac{n+2}{n}\right| \lim_{n\to\infty} |x| < 1$$

$$\Rightarrow \lim_{n\to\infty} |x| < \frac{1}{\lim_{n\to\infty} \left|\frac{1+\frac{2}{n}}{1}\right|}$$

$$\Rightarrow |x| < 1$$

So, the series is convergent for $|x| < 1$

The radius of convergence $R = 1$

Centre is 0.

23-TO-25

Shift the index so that the power under the ~~summand~~ summation sign is x^n. Also determine the radius of convergence.

24. $$\sum_{p=2}^{\infty} p(p-1)x^{p-2}$$

Put $p-2 = n$

when $p = 2, \quad n = 0$

$p = \infty, \quad n = \infty$

Thus, $\sum_{p=2}^{\infty} (n+2)(n+1)x^n$

Thus, $$\sum_{p=2}^{\infty} p(p-1)x^{p-2} = \sum_{n=0}^{\infty} (n+2)(n+1)x^n$$

We have, the n^{th} term of the series

$$a_n = (n+2)(n+1)x^n$$

$$a_{n+1} = (n+3)(n+2)x^{n+1}$$

Scanned with CamScanner

For convergence,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{(n+3)(n+2)x^{n+1}}{(n+2)(n+1)x^n} \right| < 1$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{(n+3)x}{(n+1)} \right| < 1$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{n+3}{n+1} \right| \lim_{n\to\infty} |x| < 1$$

$$\Rightarrow \lim_{n\to\infty} |x| < \frac{1}{\lim_{n\to\infty} \left| \frac{n+3}{n+1} \right|}$$

$$\Rightarrow |x| < \frac{1}{\lim_{n\to\infty} \left| \frac{1+\frac{3}{n}}{1+\frac{1}{n}} \right|}$$

$$\Rightarrow |x| < 1$$

So, the series is convergent for $|x| < 1$

The radius of convergence $R = 1$

centre is 0.

Scanned with CamScanner

4.3 Legendre's equation, Legendre polynomial $P_n(x)$

(A) We have the general form of Legendre's differential equation is

$$(1-x^2)\,y'' - 2xy' + n(n+1)\,y = 0$$

where, n is a given real number.

For diffrent values of n we can get diffrent examples of Legendre differential equations.

For $n = 1$,

$$(1-x^2)\,y'' - 2xy' + 2y = 0$$

For $n = 2$,

$$(1-x^2)\,y'' - 2xy' + 6y = 0$$

and so on.

Question-1

Find the Power series solution of Legendre differential equation.

Ans:- we have the • Legendre differential equation is

$$(1-x^2)\,y'' - 2xy' + n(n+1)\,y = 0 \quad \text{——— (1)}$$

All the points are ordinary point of equation (1) except $x = \pm 1$

Clearly $x = 0$ is an ordinary point of equation (1)

Let the power series solution be

$$y = \sum_{m=0}^{\infty} a_m x^m \quad \text{——— (2)}$$

$$y' = \sum_{m=1}^{\infty} m\,a_m\,x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1)\,a_m\,x^{m-2}$$

Scanned with CamScanner

Putting these values in equation (1)

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n\,a_n x^{n-1} + n(n+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)\,a_n x^n - 2\sum_{n=1}^{\infty} n\,a_n x^n + n(n+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow (2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots) - (2a_2 x^2 + 6a_3 x^3 + \cdots) - 2(a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots) + n(n+1)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = 0$$

$$\Rightarrow (2a_2 + n(n+1)a_0) + (6a_3 - 2a_1 + n(n+1)a_1)x + (12a_4 - 2a_2 - 4a_2 + n(n+1)a_2)x^2 + (20a_5 - 6a_3 - 6a_3 + n(n+1)a_3)x^3 + \cdots = 0$$

Equating both sides,

$$2a_2 + n(n+1)a_0 = 0 \Rightarrow a_2 = -\frac{n(n+1)}{2!}a_0$$

$$6a_3 + (n^2+n-2)a_1 = 0 \Rightarrow a_3 = -\frac{(n+2)(n-1)}{3!}a_1$$

$$12a_4 + (n^2+n-6)a_2 = 0 \Rightarrow a_4 = -\frac{(n+3)(n-2)}{12}a_2$$

$$= \frac{(n+3)(n-2)n(n+1)}{4!}a_0$$

$$20a_5 + (n^2+n-12)a_3 = 0 \Rightarrow a_5 = -\frac{(n+4)(n-3)}{20}a_3$$

$$= \frac{(n+4)(n-3)(n+2)(n-1)}{5!}a_1$$

& so on

Putting these values in equation (2) the power series solution is

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x - \frac{n(n+1)}{2!} a_0 x^2 - \frac{(n-1)(n+2)}{3!} a_1 x^3$$

$$+ \frac{(n-2)\,n\,(n+1)(n+3)}{4!} a_0 x^4 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1 x^5$$

$$- \cdots\cdots$$

$$= a_0 \left(1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)\,n(n+1)(n+3)}{4!} x^4 - \cdots\right)$$

$$+ a_1 \left(x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \cdots\right)$$

(B) An expression for Legendre polynomial $P_n(x)$ is given by the formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n, \quad n = 0, 1, 2 \cdots$$

is known as Rodrigues formula.

For $n=0$, The first Legendre polynomial $P_0(x) = 1$

$n=1$, The 2nd " " $P_1(x) = x$

$n=2$, The 3rd " " $P_2(x) = \frac{1}{2}(3x^2-1)$

$n=3$, The 4th " " $P_3(x) = \frac{1}{2}(5x^3-3x)$

Question-2

Find the value of $P_2(x)$.

Ans:- An expression for Legendre polynomial is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n, \quad n = 0, 1, 2 \cdots$$ (Rodrigues formula)

Scanned with CamScanner

For $n=2$,

$$P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2} (x^2-1)^2$$

$$= \frac{1}{8} \frac{d^2}{dx^2} (x^4 - 2x^2 + 1)$$

$$= \frac{1}{8} \frac{d}{dx} (4x^3 - 4x)$$

$$= \frac{1}{8} (12x^2 - 4)$$

$$= \frac{1}{2} (3x^2 - 1) .$$

Question-3

Express the following polynomials in terms of Legendre polynomial

(i) $5x^3 + 3x$

(ii) $10x^3 - 5x^2 + 6x - 10$

Ans to (i)

$$5x^3 + 3x = 5\left(\frac{2P_3(x) + 3x}{5}\right) + 3x$$

$$= 2P_3(x) + 6x$$

$$= 2P_3(x) + 6P_1(x)$$

Ans to (ii)

$$10x^3 - 5x^2 + 6x - 10 = 10\left(\frac{2P_3(x) + 3x}{5}\right) - 5x^2 + 6x - 10$$

$$= 4P_3(x) + 6x - 5x^2 + 6x - 10$$

$$= 4P_3(x) - 5\left(\frac{2P_2(x) + 1}{3}\right) + 12x - 10$$

$$= 4P_3(x) - \frac{10}{3} P_2(x) - \frac{5}{3} + 12x - 10$$

$$= 4P_3(x) - \frac{10}{3} P_2(x) + 12 P_1(x) - \frac{35}{3}$$

$$= 4P_3(x) - \frac{10}{3} P_2(x) + 12 P_1(x) - \frac{35}{3} P_0(x) .$$

Scanned with CamScanner

Question-4

State and Prove Rodrigues formula.

Ans:-

Statement:-

An expression for Legendre polynomial is given by the formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n, \quad n = 0, 1, 2 \cdots$$

is known as Rodrigues formula.

Proof:-

Put $V = (x^2-1)^n$

$$\frac{dV}{dx} = n(x^2-1)^{n-1} 2x$$

multiplying (x^2-1) both sides

$$(x^2-1)\frac{dV}{dx} = 2nx(x^2-1)^n$$

$$\Rightarrow -(1-x^2)\frac{dV}{dx} = 2nxV$$

$$\Rightarrow (1-x^2)\frac{dV}{dx} + 2nxV = 0$$

Again differentiating both sides w.r. to x

$$(1-x^2)\frac{d^2V}{dx^2} - 2x\frac{dV}{dx} + 2nx\frac{dV}{dx} + 2nV = 0$$

$$\Rightarrow (1-x^2)\frac{d^2V}{dx^2} + (2n-2)x\frac{dV}{dx} + 2nV = 0$$

Again differentiating n times using Lebnitz rule

Lebnitz rule

$$(UV)_n = U_n V + {}^nC_1 U_{n-1} V_1 + {}^nC_2 U_{n-2} V_2 + \cdots + UV_n$$

$$= U_n V + n U_{n-1} V_1 + \frac{n(n-1)}{2!} U_{n-2} V_2 + \cdots + UV_n$$

Ex:- $(x^2 e^x)_n$ (V U)

$$= e^x x^2 + n e^x \cdot 2x + \frac{n(n-1)}{2!} e^x 2$$

$$= x^2 e^x + 2nx e^x + n(n-1) e^x$$

Scanned with CamScanner

$$(1-x^2)\frac{d^{n+2}V}{dx^{n+2}} + n(-2x)\frac{d^{n+1}V}{dx^{n+1}} + \frac{n(n-1)}{2!}(-2)\frac{d^nV}{dx^n}$$

$$+ (2n-2)x\frac{d^{n+1}V}{dx^{n+1}} + n(2n-2)\frac{d^nV}{dx^n} + 2n\frac{d^nV}{dx^n} = 0$$

$$\Rightarrow (1-x^2)\frac{d^{n+2}V}{dx^{n+2}} - 2nx\frac{d^{n+1}V}{dx^{n+1}} - n^2\frac{d^nV}{dx^n} + n\frac{d^nV}{dx^n}$$

$$+ 2nx\frac{d^{n+1}V}{dx^{n+1}} - 2x\frac{d^{n+1}V}{dx^{n+1}} + 2n^2\frac{d^nV}{dx^n} - 2n\frac{d^nV}{dx^n} + 2n\frac{d^nV}{dx^n} = 0$$

$$\Rightarrow (1-x^2)\frac{d^{n+2}V}{dx^{n+2}} - 2x\frac{d^{n+1}V}{dx^{n+1}} + n(n+1)\frac{d^nV}{dx^n} = 0.$$

Put $V = \frac{d^nV}{dx^n}$

Thus we have,

$$(1-x^2)\frac{d^2V}{dx^2} - 2x\frac{dV}{dx} + n(n+1)V = 0$$

which shows that V satisfies Legendre differential equation and is a polynomial

So the Legendre polynomial can be written as

$$P_n(x) = cV \quad \text{---------- (1)}$$ where c is the Constant can be determined from the condition

$P_n(1) = 1$.

Now, $P_n(x) = cV = c\frac{d^nV}{dx^n} = c\frac{d^n}{dx^n}(x^2-1)^n$

$$= c\frac{d^n}{dx^n}\left[(x-1)^n(x+1)^n\right]$$

$$= c\left[n!(x+1)^n + n!(x-1)^n\right]$$

Now, $P_n(1) = 1$

$\Rightarrow c\left[n!\,2^n + n!\cdot 0\right] = 1$

$\Rightarrow c = \frac{1}{2^n n!}$

Scanned with CamScanner

Putting the value of c in equation (1)

$$P_n(x) = cV$$

$$= c\frac{d^n V}{dx^n}$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n = 0, 1, 2 \cdots\cdots,$$

(c) Orthogonalty property of Legendre polynomial

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

where $P_m(x)$ and $P_n(x)$ are Legendre polynomial of degree m and n respectively.

Question - 5

Find the value of

(i) $\int_{-1}^{1} P_1(x) P_3(x)\, dx$ (ii) $\int_{-1}^{1} P_3^2(x)\, dx$

Ans: From the orthogonal properties of Legendre polynomial

(i) $\int_{-1}^{1} P_1(x) P_3(x)\, dx = 0.$

(ii) $\int_{-1}^{1} P_3^2(x)\, dx = \frac{2}{2\times 3+1} = \frac{2}{7}.$

Question - 6

Show that for $m \neq n$

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$$

Ans:- Since $P_n(x)$ is a Legendre polynomial So it must satisfy Legendre differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Scanned with CamScanner

i.e $(1-x^2)\frac{d^2P_n(x)}{dx^2} - 2x\frac{dP_n(x)}{dx} + n(n+1)P_n(x) = 0$

$\Rightarrow \frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] = -n(n+1)P_n(x)$

Multiplying both sides by $P_m(x)$ & integrating with respect to x from $x=-1$ to $x=+1$

$$\int_{-1}^{1} P_m(x)\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right]dx = -n(n+1)\int_{-1}^{1} P_m(x)P_n(x)dx$$

Integrating by Parts

$$\left(P_m(x)(1-x^2)\frac{dP_n(x)}{dx}\right)_{-1}^{1} - \int_{-1}^{1}(1-x^2)\frac{dP_n(x)}{dx}\frac{dP_m(x)}{dx}dx = -n(n+1)\int_{-1}^{1}P_m(x)P_n(x)dx$$

$$\Rightarrow 0 + \int_{-1}^{1}(1-x^2)\frac{dP_n(x)}{dx}\cdot\frac{dP_m(x)}{dx}dx = n(n+1)\int_{-1}^{1}P_m(x)P_n(x)dx \quad \text{——(1)}$$

Similarly $P_m(x)$ is a Legendre polynomial

Continuing like this we get (Interchanging m and n)

$$\int_{-1}^{1}(1-x^2)\frac{dP_m(x)}{dx}\frac{dP_n(x)}{dx}dx = m(m+1)\int_{-1}^{1}P_n(x)P_m(x)dx \quad \text{——(2)}$$

Subtracting equation (2) from equation (1)

$$n(n+1)\int_{-1}^{1}P_m(x)P_n(x)dx - m(m+1)\int_{-1}^{1}P_m(x)P_n(x)dx = 0$$

$$\Rightarrow [n(n+1) - m(m+1)]\int_{-1}^{1}P_m(x)P_n(x)dx = 0$$

Since, $m \neq n$ So, $n(n+1) - m(m+1) \neq 0$

Thus, $\int_{-1}^{1}P_m(x)P_n(x)dx = 0$

Scanned with CamScanner

(D) The generating function for Legendre polynomial is given by,

$$\frac{1}{\sqrt{1-2xv+v^2}} = \sum_{n=0}^{\infty} P_n(x)\, v^n .$$

NOTE:-

(i) If we expand the function $\frac{1}{\sqrt{1-2xv+v^2}}$ and equate both sides we can get all the values of Legendre polynomial $P_n(x)$ for $n=0,1,2\cdots\cdots$

(ii) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$

Question-7

Show that,

(i) $P_n(1) = 1$

(ii) $P_n(-1) = (-1)^n$

Ans:-

(i) we have from the generating function for Legendre polynomial

$$\frac{1}{\sqrt{1-2xv+v^2}} = \sum_{n=0}^{\infty} P_n(x)\, v^n$$

Put $x=1$,

$$\frac{1}{\sqrt{1-2v+v^2}} = \sum_{n=0}^{\infty} P_n(1)\, v^n$$

$$\Rightarrow \frac{1}{\sqrt{(1-v)^2}} = \sum_{n=0}^{\infty} P_n(1)\, v^n$$

$$\Rightarrow (1-v)^{-1} = \sum_{n=0}^{\infty} P_n(1)\, v^n$$

$$\Rightarrow \sum_{n=0}^{\infty} v^n = \sum_{n=0}^{\infty} P_n(1)\, v^n .$$

Scanned with CamScanner

Equating the co-efficients of v^n of both sides.

$$1 = P_n(1)$$

$$\Rightarrow P_n(1) = 1.$$

(II) we have from the generating function for Legendre polynomial

$$\frac{1}{\sqrt{1-2xv+v^2}} = \sum_{n=0}^{\infty} P_n(x)\, v^n$$

Put $x=-1$,

$$\frac{1}{\sqrt{1+2v+v^2}} = \sum_{n=0}^{\infty} P_n(-1)\, v^n$$

$$\Rightarrow \frac{1}{\sqrt{(1+v)^2}} = \sum_{n=0}^{\infty} P_n(-1)\, v^n$$

$$\Rightarrow (1+v)^{-1} = \sum_{n=0}^{\infty} P_n(-1)\, v^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n v^n = \sum_{n=0}^{\infty} P_n(-1)\, v^n$$

Equating the coefficients of v^n of both sides

$$(-1)^n = P_n(-1)$$

$$\Rightarrow P_n(-1) = (-1)^n.$$

<u>Question-8</u>

show that

(i) $P_n(-x) = (-1)^n P_n(x)$

(ii) $P_n'(-x) = (-1)^{n+1} P_n'(x)$

<u>Ans:-</u>

(i)

Scanned with CamScanner

(D) The generating function for Legendre polynomial is given by,

$$\frac{1}{\sqrt{1-2xv+v^2}} = \sum_{n=0}^{\infty} P_n(x)\, v^n .$$

NOTE:-

(i) If we expand the function $\frac{1}{\sqrt{1-2xv+v^2}}$ and equate both sides we can get all the values of Legendre polynomial $P_n(x)$ for $n=0,1,2\ldots\ldots$

(II) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$

Question-7

Show that,

(i) $P_n(1) = 1$

(II) $P_n(-1) = (-1)^n$

Ans:-

(i) we have from the generating function for Legendre polynomial

$$\frac{1}{\sqrt{1-2xv+v^2}} = \sum_{n=0}^{\infty} P_n(x)\, v^n$$

Put $x=1$,

$$\frac{1}{\sqrt{1-2v+v^2}} = \sum_{n=0}^{\infty} P_n(1)\, v^n$$

$$\Rightarrow \frac{1}{\sqrt{(1-v)^2}} = \sum_{n=0}^{\infty} P_n(1)\, v^n$$

$$\Rightarrow (1-v)^{-1} = \sum_{n=0}^{\infty} P_n(1)\, v^n$$

$$\Rightarrow \sum_{n=0}^{\infty} v^n = \sum_{n=0}^{\infty} P_n(1)\, v^n .$$

Scanned with CamScanner

Equating the co-efficients of v^n of both sides.

$$1 = P_n(1)$$

$$\Rightarrow P_n(1) = 1.$$

(II) we have from the generating function for Legendre polynomial

$$\frac{1}{\sqrt{1-2xv+v^2}} = \sum_{n=0}^{\infty} P_n(x)\, v^n$$

Put $x=-1$,

$$\frac{1}{\sqrt{1+2v+v^2}} = \sum_{n=0}^{\infty} P_n(-1)\, v^n$$

$$\Rightarrow \frac{1}{\sqrt{(1+v)^2}} = \sum_{n=0}^{\infty} P_n(-1)\, v^n$$

$$\Rightarrow (1+v)^{-1} = \sum_{n=0}^{\infty} P_n(-1)\, v^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n v^n = \sum_{n=0}^{\infty} P_n(-1)\, v^n$$

Equating the coefficients of v^n of both sides

$$(-1)^n = P_n(-1)$$

$$\Rightarrow P_n(-1) = (-1)^n.$$

Question-8

show that

(i) $P_n(-x) = (-1)^n P_n(x)$

(ii) $P_n'(-x) = (-1)^{n+1} P_n'(x)$

Ans:-

(i) we have, the generating function for Legendre polynomial is given by,

Scanned with CamScanner

$$\frac{1}{\sqrt{1-2xv+v^2}} = \sum_{n=0}^{\infty} P_n(x)\, v^n \qquad \text{——— (1)}$$

Put $x = -x$,

$$\Rightarrow \frac{1}{\sqrt{1+2xv+v^2}} = \sum_{n=0}^{\infty} P_n(-x)\, v^n$$

$$\Rightarrow \frac{1}{\sqrt{1-2x(-v)+(-v)^2}} = \sum_{n=0}^{\infty} P_n(-x)\, v^n$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)\,(-v)^n = \sum_{n=0}^{\infty} P_n(-x)\, v^n. \qquad \text{(From (1))}$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n P_n(x)\, v^n = \sum_{n=0}^{\infty} P_n(-x)\, v^n.$$

Equating both sides, the co-efficients of v^n

$$(-1)^n P_n(x) = P_n(-x)$$

$$\Rightarrow P_n(-x) = (-1)^n P_n(x)$$

(II) we know that,

$$P_n(-x) = (-1)^n P_n(x)$$

Diff. both sides w.r. to x

$$(-1)\, P_n'(-x) = (-1)^n P_n'(x)$$

Multiplying both sides by (-1)

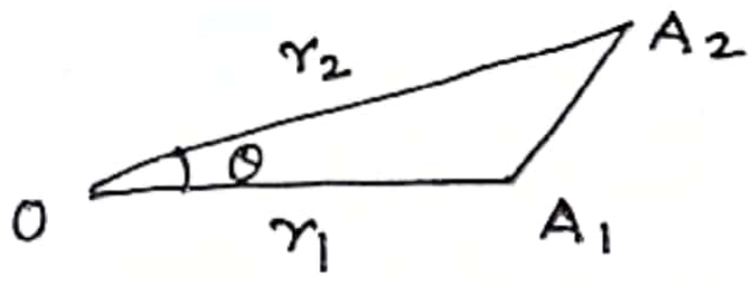
$$(-1)^2 P_n'(-x) = (-1)^{n+1} P_n'(x)$$

$$\Rightarrow P_n'(-x) = (-1)^{n+1} P_n'(x).$$

Scanned with CamScanner

Question-9

Let A_1 & A_2 be any two points in space



show that,

$$\frac{1}{r} = \frac{1}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}} = \frac{1}{r_2}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n.$$

Ans:- We have from the properties of triangle

$$r^2 = r_1^2+r_2^2-2r_1r_2\cos\theta$$

$$\Rightarrow r = \sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}$$

$$\Rightarrow \frac{1}{r} = \frac{1}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}} \quad \text{——— (1)}$$

We have from the generating function for Legendre polynomial

$$\frac{1}{\sqrt{1-2xv+v^2}} = \sum_{n=0}^{\infty} P_n(x)v^n$$

Put $x = \cos\theta$ $v = \frac{r_1}{r_2}$,

$$\frac{1}{\sqrt{1-2\frac{r_1}{r_2}\cos\theta+\left(\frac{r_1}{r_2}\right)^2}} = \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n$$

$$\Rightarrow \frac{1}{\sqrt{\frac{r_1^2+r_2^2-2r_1r_2\cos\theta}{r_2^2}}} = \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n$$

$$\Rightarrow \frac{r_2}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}} = \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n$$

$$\Rightarrow \frac{1}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}} = \frac{1}{r_2}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n \quad \text{——(2)}$$

From (1) & (2)

$$\frac{1}{r} = \frac{1}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}} = \frac{1}{r_2}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n$$

Scanned with CamScanner

Question-10

State and prove Bonnets recursion formula

Statement

$$(n+1)\,P_{n+1}(x) = (2n+1)\,x\,P_n(x) - n\,P_{n-1}(x)$$

we have from the generating function for Legendre polynomial

$$\frac{1}{\sqrt{1-2xv+v^2}} = \sum_{n=0}^{\infty} P_n(x)\,v^n$$

$$\Rightarrow \quad (1-2xv+v^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\,v^n \qquad \text{——— (1)}$$

Diff. both sides w.r. to v

$$-\frac{1}{2}(1-2xv+v^2)^{-\frac{3}{2}}(-2x+2v) = \sum_{n=1}^{\infty} n\,P_n(x)\,v^{n-1}$$

$$\Rightarrow \quad (x-v)(1-2xv+v^2)^{-\frac{3}{2}} = \sum_{n=1}^{\infty} n\,P_n(x)\,v^{n-1}$$

Multiplying both sides by $(1-2xv+v^2)$

$$\Rightarrow \quad (x-v)(1-2xv+v^2)^{-\frac{1}{2}} = (1-2xv+v^2)\sum_{n=1}^{\infty} n\,P_n(x)\,v^{n-1}$$

$$\Rightarrow \quad (x-v)\sum_{n=0}^{\infty} P_n(x)\,v^n = (1-2xv+v^2)\sum_{n=1}^{\infty} n\,P_n(x)\,v^{n-1}$$

(From (1))

Equating the co-efficients of v^n of both sides,

$$x\,P_n(x) - P_{n-1}(x) = (n+1)\,P_{n+1}(x) - 2xn\,P_n(x) + (n-1)\,P_{n-1}(x)$$

$$\Rightarrow \quad x\,P_n(x) - P_{n-1}(x) = (n+1)\,P_{n+1}(x) - 2xn\,P_n(x) + n\,P_{n-1}(x) - P_{n-1}(x)$$

$$\Rightarrow \quad (n+1)\,P_{n+1}(x) = 2nx\,P_n(x) + x\,P_n(x) - n\,P_{n-1}(x)$$

$$\Rightarrow \quad (n+1)\,P_{n+1}(x) = (2n+1)\,x\,P_n(x) - n\,P_{n-1}(x)$$

Scanned with CamScanner

$x P_n(x) v^n \rightarrow x P_n(x)$

$P_n(x) v^{n+1} \rightarrow P_{n-1}(x)$

$n P_n(x) v^{n-1} \rightarrow (n+1) P_{n+1}(x)$

$2xn P_n(x) v^n \rightarrow 2xn P_n(x)$

$n P_n(x) v^{n+1} \rightarrow (n-1) P_{n-1}(x)$

Question-11

Show that

(i) $P_{2n}(0) = (-1)^n \frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots 2n}$

(ii) $P_{2n+1}(0) = 0$

Ans to (i)

We have from the generating function for Legendre polynomial

$$\frac{1}{\sqrt{1-2xv+v^2}} = \sum_{n=0}^{\infty} P_n(x) v^n.$$

Put $x = 0$,

$$\frac{1}{\sqrt{1+v^2}} = \sum_{n=0}^{\infty} P_n(0) v^n.$$

$$\Rightarrow (1+v^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(0) v^n .$$

$$\Rightarrow 1 - \frac{1}{2}v^2 + \frac{\frac{1}{2}(\frac{1}{2}+1)}{2!}(v^2)^2 - \frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{3!}(v^2)^3$$

$$+ \cdots\cdots (-1)^r \frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)\cdots(\frac{1}{2}+r-1)}{r!} - \cdots = \sum_{n=0}^{\infty} P_n(0) v^n$$

Scanned with CamScanner

$$\Rightarrow 1 - \frac{1}{2}v^2 + \frac{1\cdot 3}{2\cdot 4}v^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}v^6 + \cdots (-1)^r \frac{1\cdot 3\cdot 5 \cdots (2r-1)}{2\cdot 4\cdot 6 \cdots 2r} v^{2r}$$

$$- \cdots = \sum_{n=0}^{\infty} P_n(0)\, v^n$$

Equating the co-efficient of v^{2n} of both sides

$$(-1)^n \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} = P_{2n}(0)$$

$$\Rightarrow P_{2n}(0) = (-1)^n \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n}$$

(ii) Equating the co-efficient of v^{2n+1} of both sides.

$$P_{2n+1}(0) = 0$$

NOTE:-

$$(1+x)^{-m} = 1 - mx + \frac{m(m+1)}{2!}x^2 - \frac{m(m+1)(m+2)}{3!}x^3 ,$$

$$+ \cdots (-1)^r \frac{m(m+1)(m+2)\cdots(m-r+1)}{r!}x^r - \cdots$$

Scanned with CamScanner

4.5 Bessel's equation, Bessel Function $J_v(x)$

We have the general form of Bessel's differential equation is

$$x^2y'' + xy' + (x^2 - v^2)y = 0 \quad \text{——— (1)}$$

where the parameter v is a given number. For diffrent values of v we can get diffrent examples of Bessels DE.

For $v = 2$, $x^2y'' + xy' + (x^2 - 4)y = 0$

For $v = \frac{1}{3}$, $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$

The general solution of equation (1) is given by

$$\boxed{y = C_1 J_v(x) + C_2 J_{-v}(x)}$$

where,

$$J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(v+m+1)} \left(\frac{x}{2}\right)^{v+2m}$$

$$J_{-v}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(-v+m+1)} \left(\frac{x}{2}\right)^{-v+2m}$$

Properties of Γ function

(i) Γ(-ve no., zero) $\rightarrow$ does not exist

(ii) $\Gamma 1 = 1$ (iii) $\Gamma\frac{1}{2} = \sqrt{\pi}$

(iv) $\Gamma n = (n-1)\,\Gamma(n-1) = (n-1)(n-2)\,\Gamma(n-2)$

$= (n-1)(n-2)(n-3)\,\Gamma(n-3)$

$= (n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1\,\Gamma 1$

$= (n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1\cdot 1$

$= (n-1)!$

Scanned with CamScanner

Question:-1

Find the general solution of

$$x^2y'' + xy' + (x^2 - \tfrac{1}{9})y = 0 .$$

Ans:- which is a Bessels differential equation with $v = \frac{1}{3}$.

Thus the general solution

$$y = C_1 J_{\frac{1}{3}}(x) + C_2 J_{-\frac{1}{3}}(x) .$$

Question:-2

show that,

$$\frac{d}{dx}\left[x^v J_v(x) \right] = x^v J_{v-1}(x) .$$

LHS

$$\frac{d}{dx}\left[x^v J_v(x) \right]$$

$$= \frac{d}{dx}\left[x^v \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(v+m+1)} \left(\frac{x}{2}\right)^{v+2m} \right]$$

$$= \frac{d}{dx}\left[\sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2v+2m}}{m!\,\Gamma(v+m+1)\, 2^{v+2m}} \right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m\,(2v+2m)\, x^{2v+2m-1}}{m!\,\Gamma(v+m+1)\, 2^{v+2m}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m\, 2(v+m)\, x^{2v+2m-1}}{m!\,(v+m)\,\Gamma(v+m)\, 2^{v+2m}}$$

$$= x^v \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{v-1+2m}}{m!\,\Gamma(v-1+m+1)\, 2^{v-1+2m}}$$

$$= x^v \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(v-1+m+1)} \left(\frac{x}{2}\right)^{v-1+2m}$$

$$= x^v J_{v-1}(x)$$

Scanned with CamScanner

Question-3

Show that,

$$\frac{d}{dx}\left[x^{-v} J_v(x)\right] = -x^{-v} J_{v+1}(x)$$

Ans:-

LHS:- $\frac{d}{dx}\left[x^{-v} J_v(x)\right]$

$$= \frac{d}{dx}\left[x^{-v} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(v+m+1)} \left(\frac{x}{2}\right)^{v+2m}\right]$$

$$= \frac{d}{dx}\left[\sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{m!\,\Gamma(v+m+1)\, 2^{v+2m}}\right]$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m\, 2m\, x^{2m-1}}{m!\,\Gamma(v+m+1)\, 2^{v+2m}}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m\, x^{2m-1}}{(m-1)!\,\Gamma(v+m+1)\, 2^{v-1+2m}}$$

$$= -x^{-v} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\, x^{v-1+2m}}{(m-1)!\,\Gamma(v+m+1)\, 2^{v-1+2m}}$$

Put $m-1 = n$

when $m=1$, $n=0$

$m=\infty$, $n=\infty$

$$= -x^{-v} \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{v-1+2(n+1)}}{n!\,\Gamma(v+n+1+1)\, 2^{v-1+2(n+1)}}$$

$$= -x^{-v} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(v+1+n+1)} \left(\frac{x}{2}\right)^{v+1+2n}$$

$$= -x^{-v} J_{v+1}(x) \qquad \text{RHS.}$$

Scanned with CamScanner

Question - 4

Show that,

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$$

Ans:- we know that,

$$\frac{d}{dx}\left[x^v J_v(x) \right] = x^v J_{v-1}(x)$$

$$\Rightarrow x^v J_v'(x) + v x^{v-1} J_v(x) = x^v J_{v-1}(x)$$

$$\Rightarrow J_v'(x) + \frac{v}{x} J_v(x) = J_{v-1}(x) \quad \text{——— (1)}$$

Again we have,

$$\frac{d}{dx}\left[x^{-v} J_v(x) \right] = -x^{-v} J_{v+1}(x)$$

$$\Rightarrow x^{-v} J_v'(x) - v x^{-v-1} J_v(x) = -x^{-v} J_{v+1}(x)$$

$$\Rightarrow J_v'(x) - \frac{v}{x} J_v(x) = -J_{v+1}(x) \quad \text{——— (2)}$$

Subtracting equation (2) from equation (1)

$$J_v'(x) + \frac{v}{x} J_v(x) = J_{v-1}(x)$$

$$\underline{-\ J_v'(x) \ -\ \frac{v}{x} J_v(x) = -J_{v+1}(x)}$$

$$\frac{2v}{x} J_v(x) = J_{v-1}(x) + J_{v+1}(x)$$

$$\Rightarrow J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x).$$

Question - 5

show that,

$$J_{v-1}(x) - J_{v+1}(x) = 2 J_v'(x)$$

Scanned with CamScanner

Ans:- As in question number - 4 find out equation (1) & (2)

Adding equation (1) & (2)

$$J_v'(x) + \frac{v}{x} J_v(x) = J_{v-1}(x)$$

$$J_v'(x) - \frac{v}{x} J_v(x) = -J_{v+1}(x)$$

$$2J_v'(x) = J_{v-1}(x) - J_{v+1}(x)$$

$$\Rightarrow J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$$

Question-6

show that $J_0'(x) = -J_1(x)$

Ans:- we know that

$$\frac{d}{dx}\left[x^{-v} J_v(x)\right] = -x^{-v} J_{v+1}(x)$$

put $v = 0$,

$$\frac{d}{dx}\left[x^{-0} J_0(x)\right] = -x^{-0} J_{0+1}(x)$$

$$\Rightarrow \frac{d}{dx}\left[J_0(x)\right] = -J_1(x)$$

$$\Rightarrow J_0'(x) = -J_1(x)$$

Question-7

show that,

$$J_2'(x) = \frac{1}{2}\left[J_1(x) - J_3(x)\right]$$

Scanned with CamScanner

Ans:- we know that,

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$$

$$\Rightarrow J_v'(x) = \frac{1}{2}[J_{v-1}(x) - J_{v+1}(x)]$$

put $v = 2$,

$$J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$$

Question-8

show that,

$$J_1'(x) = J_0(x) - x^{-1}J_1(x)$$

Ans:- we know that,

$$\frac{d}{dx}[x^v J_v(x)] = x^v J_{v-1}(x)$$

$$\Rightarrow x^v J_v'(x) - v x^{v-1} J_v(x) = x^v J_{v-1}(x)$$

Dividing both sides by x^v

$$J_v'(x) - \frac{v}{x} J_v(x) = J_{v-1}(x)$$

put $v = 1$,

$$J_1'(x) - \frac{1}{x} J_1(x) = J_0(x)$$

$$\Rightarrow J_1'(x) = J_0(x) - \frac{1}{x} J_1(x)$$

$$\Rightarrow J_1'(x) = J_0(x) - x^{-1} J_1(x) .$$

Scanned with CamScanner

Question-9

show that,

$$\int x^{v} J_{v-1}(x)\,dx = x^{v} J_{v}(x) + C.$$

Ans:- we know that,

$$\frac{d}{dx}\left[x^{v} J_{v}(x)\right] = x^{v} J_{v-1}(x)$$

$$\Rightarrow d\left[x^{v} J_{v}(x)\right] = x^{v} J_{v-1}(x)\,dx$$

Integrating both sides,

$$\int x^{v} J_{v-1}(x)\,dx = \int d\left[x^{v} J_{v}(x)\right]$$

$$\Rightarrow \int x^{v} J_{v-1}(x)\,dx = x^{v} J_{v}(x) + C.$$

Question-10

Show that,

$$\int x^{-v} J_{v+1}(x)\,dx = -x^{-v} J_{v}(x) + C.$$

Ans:. we know that,

$$\frac{d}{dx}\left[x^{-v} J_{v}(x)\right] = -x^{-v} J_{v+1}(x)$$

$$\Rightarrow d\left[x^{-v} J_{v}(x)\right] = -x^{-v} J_{v+1}(x)$$

$$\Rightarrow x^{-v} J_{v+1}(x)\,dx = -d\left[x^{-v} J_{v}(x)\right]$$

Integrating both sides,

$$\int x^{-v} J_{v+1}(x)\,dx = -\int d\left[x^{-v} J_{v}(x)\right]$$

$$\Rightarrow \int x^{-v} J_{v+1}(x)\,dx = -x^{-v} J_{v}(x) + C.$$

Scanned with CamScanner

Question-11

Show that,

$$\int J_{v+1}(x)\,dx = \int J_{v-1}(x)\,dx - 2J_v(x)$$

Ans:- we know that,

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$$

$$\Rightarrow J_{v+1}(x) = J_{v-1}(x) - 2J_v(x)$$

$$\Rightarrow J_{v+1}(x) = J_{v-1}(x) - 2\frac{d}{dx}J_v(x)$$

$$\Rightarrow J_{v+1}(x)\,dx = J_{v-1}(x)\,dx - 2\,d\,J_v(x)$$

Integrating both sides,

$$\int J_{v+1}(x)\,dx = \int J_{v-1}(x)\,dx - 2\int d\,J_v(x)$$

$$\Rightarrow \int J_{v+1}(x)\,dx = \int J_{v-1}(x)\,dx - 2J_v(x).$$

Scanned with CamScanner

Question-12

Find $\int J_3(x)\,dx$

Ans:-

$$\int J_3(x)\,dx$$

$$= \int 1\, J_3(x)\,dx$$

$$= \int x^2 x^{-2} J_3(x)\,dx$$

$$= x^2 \int x^{-2} J_3(x)\,dx - \int \left[\int x^{-2} J_3(x)\,dx\right] 2x\,dx$$

$$= x^2 \left[-x^{-2} J_2(x)\right] - \int -x^{-2} J_2(x) \cdot 2x\,dx$$

$$= -J_2(x) + 2\int x^{-1} J_2(x)\,dx$$

$$= -J_2(x) + 2\left[-x^{-1} J_1(x)\right] + C$$

$$= -J_2(x) - 2x^{-1} J_1(x) + C$$

Question-13

Find $\int x^3 J_0(x)\,dx$

Ans:

$$\int x^3 J_0(x)\,dx$$

$$= \int x^2 x\, J_0(x)\,dx$$

$$= x^2 \int x\, J_0(x)\,dx - \int\left[\int x\, J_0(x)\,dx\right] 2x\,dx$$

$$= x^2 \left[x J_1(x)\right] - 2\int x J_1(x) \cdot x\,dx$$

$$= x^3 J_1(x) - 2\int x^2 J_1(x)\,dx$$

$$= x^3 J_1(x) - 2\left[x^2 J_2(x)\right] + C$$

$$= x^3 J_1(x) - 2x^2 J_2(x) + C\,.$$

Scanned with CamScanner

Question-14

Find $\int J_5(x)\,dx$

Ans:-

$$\int J_5(x)\,dx$$

$$= \int 1\, J_5(x)\,dx$$

$$= \int x^4 x^{-4} J_5(x)\,dx$$

$$= x^4 \int x^{-4} J_5(x)\,dx - \int\left[\int x^{-4} J_5(x)\,dx\right] 4x^3\,dx$$

$$= x^4\left[-x^{-4} J_4(x)\right] - \int -x^{-4} J_4(x)\cdot 4x^3\,dx$$

$$= -J_4(x) + 4\int x^{-1} J_4(x)\,dx$$

$$= -J_4(x) + 4\int x^2 x^{-3} J_4(x)\,dx$$

$$= -J_4(x) + 4\left[x^2\int x^{-3} J_4(x)\,dx - \int\left[\int x^{-3} J_4(x)\,dx\right]2x\,dx\right]$$

$$= -J_4(x) + 4\left[x^2\left(-x^{-3} J_3(x)\right) - \int -x^{-3} J_3(x)\cdot 2x\,dx\right]$$

$$= -J_4(x) - 4x^{-1} J_3(x) + 2\int x^{-2} J_3(x)\,dx$$

$$= -J_4(x) - 4x^{-1} J_3(x) + 2\left(-x^{-2} J_2(x)\right) + C$$

$$= -J_4(x) - 4x^{-1} J_3(x) - 2x^{-2} J_2(x) + C.$$

$$\left[\int x^{-v} J_{v+1}(x)\,dx = -x^{-v} J_v(x) + C\right]$$

Scanned with CamScanner

Question -15

Show that,

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

Ans:- We have,

$$J_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\frac{1}{2}+n+1)} \left(\frac{x}{2}\right)^{\frac{1}{2}+2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{x}\, x^{2n}}{n!\,\Gamma(n+\frac{3}{2})\sqrt{2}\, 2^{2n}}$$

Multiplying $\sqrt{2}\sqrt{x}$ in the N^r and D^r

$$J_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{2}\, x\, x^{2n}}{n!\,\Gamma(n+\frac{3}{2})\sqrt{x}\, 2\, 2^{2n}}$$

$$= \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,\Gamma(n+\frac{3}{2})\, 2^{2n+1}} \quad \text{——— (1)}$$

We have,

$$\Gamma(n+\tfrac{3}{2}) = (n+\tfrac{1}{2})\,\Gamma(n+\tfrac{1}{2})$$

$$= (n+\tfrac{1}{2})(n-\tfrac{1}{2})\,\Gamma(n-\tfrac{1}{2})$$

$$= (n+\tfrac{1}{2})(n-\tfrac{1}{2})(n-\tfrac{3}{2})\,\Gamma(n-\tfrac{3}{2})$$

$$= (n+\tfrac{1}{2})(n-\tfrac{1}{2})(n-\tfrac{3}{2}) \cdots \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\,\Gamma\tfrac{1}{2}$$

$$= \frac{2n+1}{2}\cdot\frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$= \frac{(2n+1)(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1\sqrt{\pi}}{2^{n+1}}$$

$$= \frac{(2n+1)\,2n\,(2n-1)(2n-2)(2n-3)\cdots 5\cdot 4\cdot 3\cdot 2\cdot 1\sqrt{\pi}}{2^{n+1}\,2n(2n-2)(2n-4)\cdots 6\cdot 4\cdot 2}$$

Scanned with CamScanner

$$= \frac{(2n+1)!\sqrt{\pi}}{2^{n+1} \cdot 2^n \, n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}$$

$$= \frac{(2n+1)!\sqrt{\pi}}{2^{2n+1} \, n!}$$

Putting these values in equation (1)

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \quad 2^{2n+1} \, n!}{n! \; 2^{2n+1} \; (2n+1)! \sqrt{\pi}}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \sqrt{\frac{2}{\pi x}} \sin x$$

Question – 16

show that,

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Ans:- we have,

$$J_{-\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \; \Gamma(-\frac{1}{2}+n+1)} \left(\frac{x}{2}\right)^{-\frac{1}{2}+2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \quad \sqrt{2} \quad x^{2n}}{n! \; \Gamma(n+\frac{1}{2}) \sqrt{x} \; 2^{2n}}$$

$$= \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n}}{n! \; \Gamma(n+\frac{1}{2}) \, 2^{2n}} \quad \text{——— (1)}$$

Scanned with CamScanner

We have,

$$\Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})\,\Gamma(n-\tfrac{1}{2})$$

$$= (n-\tfrac{1}{2})(n-\tfrac{3}{2})\,\Gamma(n-\tfrac{3}{2})$$

$$= (n-\tfrac{1}{2})(n-\tfrac{3}{2})\cdots\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\,\Gamma\tfrac{1}{2}$$

$$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$= \frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1\sqrt{\pi}}{2^n}$$

$$= \frac{2n(2n-1)(2n-2)(2n-3)\cdots 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\sqrt{\pi}}{2^n\; 2n(2n-2)(2n-4)\cdots 6\cdot 4\cdot 2}$$

$$= \frac{(2n)!\sqrt{\pi}}{2^n\; 2^n\, n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}$$

$$= \frac{(2n)!\sqrt{\pi}}{2^{2n}\; n!}$$

Putting it in equation (1)

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{x}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}\; 2^{2n}\, n!}{n!\; 2^{2n}\; (2n)!\sqrt{\pi}}$$

$$= \sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$= \sqrt{\frac{2}{\pi x}}\cos x \quad .$$

Scanned with CamScanner

Question-17

show that

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

Ans:- we know that,

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$$

Put $v = \frac{1}{2}$,

$$J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x)$$

$$\Rightarrow J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\cdot\left(\frac{\sin x}{x} - \cos x\right)$$

Question -18

show that,

$$J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

Ans:- we know that,

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$$

Put $v = -\frac{1}{2}$.

$$J_{-\frac{3}{2}}(x) + J_{\frac{1}{2}}(x) = -\frac{1}{x} J_{-\frac{1}{2}}(x)$$

$$\Rightarrow J_{-\frac{3}{2}}(x) = -\frac{1}{x} J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

Scanned with CamScanner